Correction induced by irrelevant operators in the correlators of the two-dimensional Ising model in a magnetic field

2001 J. Phys. A: Math. Gen. 34 8733

(http://iopscience.iop.org/0305-4470/34/42/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8733–8750 PII: S0305-4470(01)24173-0

Correction induced by irrelevant operators in the correlators of the two-dimensional Ising model in a magnetic field

M Caselle[1], P Grinza[1] and N Magnoli[2]

[1] Dipartimento di Fisica Teorica dell'Università di Torino and Istituto Nazionale di Fisica Nucleare, Sezione di Torino via P Giuria 1, I-10125 Torino, Italy
[2] Dipartimento di Fisica, Università di Genova and Istituto Nazionale di Fisica Nucleare, Sezione di Genova via Dodecaneso 33, I-16146 Genova, Italy

E-mail: caselle@to.infn.it, grinza@to.infn.it and magnoli@ge.infn.it

Received 17 April 2001, in final form 1 August 2001
Published 12 October 2001
Online at stacks.iop.org/JPhysA/34/8733

Abstract
We investigate the presence of irrelevant operators in the two-dimensional Ising model perturbed by a magnetic field, by studying the corrections induced by these operators in the spin–spin correlator of the model. To this end we perform a set of high-precision simulations for the correlator both along the axes and along the diagonal of the lattice. By comparing the numerical results with the predictions of a perturbative expansion around the critical point we find unambiguous evidence of the presence of such irrelevant operators. It turns out that among the irrelevant operators the one which gives the largest correction is the spin-4 operator $T^2 + \bar{T}^2$, which accounts for the breaking of the rotational invariance due to the lattice. This result agrees with what was already known for the correlator evaluated exactly at the critical point and also with recent results obtained in the case of the thermal perturbation of the model.

PACS numbers: 05.50.+q, 64.60.Fr, 75.40.-s

1. Introduction

Since the seminal work of Belavin *et al* [1] much progress has been made in the description of two-dimensional statistical models at the critical point. In particular in all the cases in which the critical point theory can be identified with one of the so called 'minimal models' a complete list of all the operators of the theories can be constructed. Moreover, it is possible to write differential equations for the correlators and in some cases find their explicit expression in terms of special functions. Following these remarkable results, in recent years much effort has been devoted to extending them also outside the critical point. This can be achieved by perturbing the CFT with one (or more) of its relevant operators. Much less is known in this case. The most important result in this context is that for some particular choices of CFTs and relevant operators these perturbations give rise to integrable models [2, 3]. In these cases again we have a rather precise description of the theory. In particular it is possible to obtain the exact asymptotic expression for the large-distance behaviour of the correlators [4]. From this information several important results (and in particular all the universal amplitude ratios) can be obtained.

Similar efforts have also been devoted to trying to study vacuum expectation values, amplitude ratios and correlators involving irrelevant operators. However progress in this direction is limited by the lack of results (both numerical and analytical) from actual statistical mechanics realizations of the models.

The only notable exception to this state of the art is the two-dimensional Ising model. In this case, thanks to a set of remarkable results on the two- and four-point correlators [5–7] several interesting results on the contribution of irrelevant operators have been obtained both in the case of the model at its critical point [8] and for $T \neq T_c$ [9, 10] (i.e. for the theory perturbed by the energy operator). Very recently these results have been used to study up to very high orders the contribution of irrelevant operators to the magnetic susceptibility for $T \neq T_c$ [10]. In this paper we shall try to extend this analysis to the case of the magnetic perturbation of the Ising model. In particular, we shall look at the contributions due to irrelevant operators to the spin–spin correlator in the presence of an external magnetic field. We shall mainly concentrate on the energy–momentum tensor $T\bar{T}$, and the operator $T^2+\bar{T}^2$ (which appear as a consequence of the breaking due to the lattice of the full rotational symmetry) which are the most important (i.e. those with smallest (in modulus) renormalization group eigenvalue) among the irrelevant operators.

In this case there is no exact expression for the correlators or for the free energy of the model and the only way that we have to judge the validity of the CFT predictions is to compare them with the results of numerical simulations. Notwithstanding this there is a number of good reasons to choose exactly this model and this observable. Let us look at them in detail.

- The Ising model in a magnetic field is known to be exactly integrable [3], thus, even if it is not exactly solvable (although exact expressions for the free energy exist for the spin-1 Ising model in a magnetic field [11]), much important information can all the same be obtained; in particular very precise large-distance expansions exist for the correlators and the vacuum expectation values of the relevant operators are known in the continuum limit [12].
- The model is an optimal choice from the point of view of numerical simulations, since very fast and efficient algorithms exist to study it.
- The model can also be realized as a particular case of the so called IRF (interaction round a face) models [13]. In this framework several interesting results can be obtained on the spectrum of the model [14].
- By looking at the spin–spin correlator instead of the susceptibility we may study the effects of the non-zero spin operators which appear as a consequence of the breaking of the rotational symmetry due to the lattice (see below for a precise definition). These operators also appear in rotational invariant quantities like the susceptibility, but only at the second order [15, 16], i.e. at such a high power of the perturbing constant that they can be observed only if the exact solution of the model is known (as in the papers [9, 10]) and cannot be seen if one can only resort to numerical simulations.
- A very interesting result of [10] (which had already been anticipated 20 years ago by Aharony and Fisher [17] and then observed in [18] for the one-dimensional Ising quantum chain and in [19] for the two-dimensional square lattice Ising model) is that in the thermal

perturbation of the Ising model the energy–momentum tensor $T\bar{T}$ seems to be absent. The first correction which involves irrelevant operators is thus due to the spin-4 operator $T^2 + \bar{T}^2$. This result should also hold at the critical point, where it can be independently observed by using transfer matrix methods [15]. However, apparently, it disagrees with the explicit form of the correlator at the critical point (which is exactly known on the lattice), which contains a scalar correction which can only be interpreted as due to the $T\bar{T}$ term. One of the aims of the present paper is to understand this puzzle.

- If one is interested in studying irrelevant operators it is mandatory to look at the short-distance behaviour of the correlator. In this respect the natural framework in which one must operate is the so called IRS expansion [20, 21]. This approach has been recently discussed in great detail [22] exactly in the case of the Ising model in a magnetic field in which we are interested here. It is only by using the results of [22] as the input of our analysis that we shall be able to reach the high level of precision which is needed in order to observe the very small corrections which are the signatures of the irrelevant operators.

This paper is organized as follows. In section 2 we shall briefly summarize some known results on the two-dimensional Ising model in a magnetic field. In section 3 we shall discuss the most important contributions due to the irrelevant operators to the spin–spin correlators and shall evaluate their magnitude and behaviour. In section 4 we shall present the numerical simulations that we have performed and finally in section 5 we shall discuss the comparison between numerical results and theoretical predictions. Section 6 is devoted to some concluding remarks.

2. The two-dimensional Ising model in a magnetic field

We shall be interested in the following in the Ising model defined on a two-dimensional square lattice of size L with periodic boundary conditions, in the presence of an external magnetic field H. The model is defined by the following partition function:

$$Z = \sum_{\sigma_i=\pm 1} \mathrm{e}^{\beta(\sum_{\langle i,j\rangle} \sigma_i\sigma_j + H\sum_i \sigma_i)} \tag{1}$$

where the notation $\langle i, j\rangle$ denotes nearest-neighbour sites in the lattice. In order to select only the magnetic perturbation, β must be fixed to its critical value:

$$\beta = \beta_{\mathrm{c}} = \tfrac{1}{2}\log(\sqrt{2}+1) = 0.440\,6868\ldots.$$

By defining $h_l = \beta_{\mathrm{c}} H$ we have

$$Z = \sum_{\sigma_i=\pm 1} \mathrm{e}^{\beta_{\mathrm{c}}\sum_{\langle i,j\rangle} \sigma_i\sigma_j + h_l\sum_i \sigma_i}. \tag{2}$$

The magnetization $M(h)$ is defined as usual

$$M(h) \equiv \frac{1}{N}\frac{\partial}{\partial h_l}(\log\ Z)|_{\beta=\beta_{\mathrm{c}}} = \left\langle \frac{1}{N}\sum_i \sigma_i \right\rangle \tag{3}$$

where $N \equiv L^2$ denotes the number of sites of the lattice.

Equation (2) is the typical partition function of a perturbed critical model. With the choice $\beta = \beta_{\mathrm{c}}$ the only perturbing operator is

$$\sigma_l \equiv \frac{1}{N}\sum_i \sigma_i. \tag{4}$$

We shall call σ_l the spin operator in the following (more precisely the lattice discretization of the spin operator). Notice that the mean value of σ_l coincides with $M(h)$:

$$\langle \sigma_l \rangle \equiv M(h). \tag{5}$$

Our goal in this paper is to study the contribution of the irrelevant operators to the spin–spin correlator. To this end we shall first study the model at the critical point (section 2.1); we shall then switch on the magnetic field (see section 2.2) and discuss the modifications that it induces in the spin–spin correlator.

2.1. The Ising model at the critical point

2.1.1. Operator content. The Ising model at the critical point is described by the unitary minimal CFT with central charge $c = 1/2$ [1]. Its spectrum can be divided into three conformal families characterized by different transformation properties under the dual and Z_2 symmetries of the model. They are the identity, spin and energy families and are commonly denoted as $[I]$, $[\sigma]$, $[\epsilon]$. Each family contains a 'primary' field (which gives the name to the entire family) and an infinite tower of 'secondary' fields (see below). The conformal weights of the primary operators are $h_I = 0$, $h_\sigma = 1/16$ and $h_\epsilon = 1/2$ respectively. Thus we see that in the Ising model the set of primary fields coincides with that of the relevant operators of the spectrum (remember that the relationship between conformal weights and renormalization group eigenvalues is $y = 2 - 2h$). This is a peculiar feature of the Ising model only, and is not shared by any other minimal unitary model. Thus in this case the irrelevant operators are bound to be secondary fields. Since in this paper we are particularly interested in the irrelevant operators, let us study in more detail the structure of the three conformal families.

- *Secondary fields.*
  All the secondary fields are generated from the primary ones by applying the generators L_{-i} and $\bar{L}_{-i}$ of the Virasoro algebra. In the following we shall denote the most general irrelevant operators in the $[\sigma]$ family (which are odd with respect to the Z_2 symmetry) by the notation σ_i and the most general operators belonging to the energy $[\epsilon]$ or to the identity $[I]$ families (which are Z_2 even) by ϵ_i and η_i respectively. It can be shown that, by applying a generator of index k: L_{-k} or $\bar{L}_{-k}$ to a field ϕ (where $\phi = I, \epsilon, \sigma$ depending on the case), of conformal weight h_ϕ, a new operator of weight $h = h_\phi + k$ is obtained. In general any combination of L_{-i} and $\bar{L}_{-i}$ generators is allowed, and the conformal weight of the resulting operator will be shifted by the sum of the indices of the generators used to create it. If we denote by n the sum of the indices of the generators of type L_{-i} and by $\bar{n}$ the sum of those of type $\bar{L}_{-i}$, the conformal weight of the resulting operator will be $h_\phi + n + \bar{n}$. The corresponding RG eigenvalue will be $y = 2 - 2h_\phi - n - \bar{n}$.
- *Non-zero spin states.*
  The secondary fields may have a non-zero spin, which is given by the difference $n - \bar{n}$. In general one is interested in scalar quantities and hence in the subset of those irrelevant operators which have $n = \bar{n}$. However, on a square lattice the rotation group is broken down to the finite dihedral subgroup D_4. Accordingly, only spin 0, 1, 2, 3 are allowed on the lattice. If an operator ϕ of the continuum theory has spin $j \in \boldsymbol{N}$, then its lattice discretization ϕ_l behaves as a spin j (mod 4) operator with respect to the C_4 subgroup. As a consequence all the operators which in the continuum limit have spin $j = 4N$ with N a non-negative integer can appear in the lattice discretization of a scalar operator.
- *Null vectors.*
  Some of the secondary fields disappear from the spectrum due to the null vector conditions. This happens in particular for one of the two states at level 2 in the σ and ϵ families and

for the unique state at level 1 in the identity family. From each null state one can generate, by applying the Virasoro operators, a whole family of null states, hence at level 2 in the identity family there is only one surviving secondary field, which can be identified with the stress energy tensor T (or $\bar{T}$).

- *Secondary fields generated by* L_{-1}.
  Among all the secondary fields a particular role is played by those generated by the L_{-1} Virasoro generator. L_{-1} is the generator of translations on the lattice and as a consequence it has zero eigenvalue on translational invariant observables.
- *Quasiprimary fields*
  A quasiprimary field $|Q\rangle$ is a secondary field which is not a null vector (or a descendent of a null vector) and satisfies the equation

$$L_1|Q\rangle = 0. \tag{6}$$

These fields play a central role in our analysis since they are the only possible candidates to be irrelevant operators of the model[3].

By imposing equation (6) it is easy to construct the first few quasiprimary operators for each conformal family. For our analysis, however, the two lowest ones are enough. They both belong to the conformal family of the identity. Their expression in terms of Virasoro generators is

$$Q_2^{\mathbf{1}} = L_{-2}|\mathbf{1}\rangle \tag{7}$$

$$Q_4^{\mathbf{1}} = (L_{-2}^2 - \tfrac{3}{5}L_{-4})|\mathbf{1}\rangle \tag{8}$$

(we use the notation Q_n^η to denote the quasiprimary state at level n in the η family).

From these fields we can construct two irrelevant operators, which both have conformal weight 4 and RG eigenvalue -2.

(1) The first combination is $Q_2^{\mathbf{1}}\bar{Q}_2^{\mathbf{1}}$, which has spin zero and can be identified with the energy–momentum tensor $T\bar{T}$.

(2) The second combination is $Q_4^{\mathbf{1}} + \bar{Q}_4^{\mathbf{1}}$, which has spin 4 and can be identified with the combination $T^2 + \bar{T}^2$.

This last contribution appears as a consequence of the breaking of the full rotational invariance due to the lattice[4].

2.1.2. Structure constants. Once the operator content is known, the only remaining information which is needed to completely identify the theory is the OPE constants. The OPE algebra is defined as

$$\Phi_i(r)\Phi_j(0) = \sum_{\{k\}} C_{ij}^{\{k\}}(r)\Phi_{\{k\}}(0) \tag{9}$$

[3] The reason is that non-quasiprimary fields can always be rewritten (using, if needed, the null vector identities) as L_{-1} acting on some combination of Virasoro generators, and thus their contribution in the perturbed action is zero. Notice however that this does not mean that these states do not exist; it is only when they are integrated over the two-dimensional plane that they give a vanishing contribution. Indeed, if we look at the model in the transfer matrix framework, we see that the spectrum also contains non-quasiprimary states. This fact may have an important consequence if one looks at the Ising quantum chain model (which belongs to the same universality class as the two-dimensional Ising model). In this case it has been claimed (see for instance [18]) that non-quasiprimary fields also contribute to the scaling function. For an updated review on quantum spin chains, and their CFT description, see [23].

[4] Notice that if we were interested in scalar quantities this term would disappear even on the lattice at first order and could contribute only at second order. This is the case for instance of the susceptibility recently discussed in [10], in which this operator gives a contribution only at order t^4 and not t^2. However in the present case since we are interested in a correlator, which defines a preferential direction on the lattice, this term can contribute already at first order. This fact will be discussed in detail in section 2.1.5 below.

where with the notation $\{k\}$ we mean that the sum runs over all the fields of the conformal family $[k]$. The structure functions $C_{ij}^k(r)$ are c-number functions of r, which must be single valued in order to take into account locality. In the large-r limit they decay with a power-like behaviour

$$C_{ij}^k(r) \sim |r|^{-\dim(C_{ij}^k)} \tag{10}$$

whose amplitude is given by

$$\hat{C}_{ij}^k \equiv \lim_{r\to\infty} C_{ij}^k(r)\ |r|^{\dim(C_{ij}^k)}. \tag{11}$$

Several of these structure constants are zero for symmetry reasons. These constraints are encoded in the so-called 'fusion rule algebra', which, in the case of the Ising model, is

$$\begin{aligned}
&[\epsilon][\epsilon] = [\mathbf{1}]\\
&[\sigma][\epsilon] = [\sigma]\\
&[\sigma][\sigma] = [\mathbf{1}] + [\epsilon].
\end{aligned} \tag{12}$$

By looking at the fusion rule algebra we can see by inspection which are the non-vanishing structure constants.

The actual value of these constants depends on the normalization of the fields, which can be chosen by fixing the long-distance behaviour of, for instance, the $\sigma\sigma$ and $\epsilon\epsilon$ correlators. In this paper we follow the commonly adopted convention, which is

$$\langle\sigma(x)\sigma(0)\rangle = \frac{1}{|x|^{\frac{1}{4}}} \qquad |x| \to \infty \tag{13}$$

$$\langle\epsilon(x)\epsilon(0)\rangle = \frac{1}{|x|^2} \qquad |x| \to \infty. \tag{14}$$

With these conventions we have, for the structure constants among primary fields,

$$\hat{C}_{\sigma,\sigma}^{\sigma} = \hat{C}_{\epsilon,\epsilon}^{\sigma} = \hat{C}_{\epsilon,\sigma}^{\epsilon} = 0 \tag{15}$$

$$\hat{C}_{\sigma,\sigma}^{\mathbf{1}} = \hat{C}_{\sigma,\mathbf{1}}^{\sigma} = \hat{C}_{\epsilon,\epsilon}^{\mathbf{1}} = \hat{C}_{\epsilon,\mathbf{1}}^{\epsilon} = 1 \tag{16}$$

and

$$\hat{C}_{\sigma,\epsilon}^{\sigma} = \hat{C}_{\sigma,\sigma}^{\epsilon} = \tfrac{1}{2}. \tag{17}$$

2.1.3. Continuum versus lattice operators. Our main interest in this paper is the spin–spin correlator on the lattice. This raises the question of the relationship between the lattice and the continuum definitions of the operator σ. In the following we shall denote the lattice discretization of the operators with the index l. Thus σ denotes the continuum operator and σ_l the lattice one.

In general the lattice operator is the most general combination of continuum operators compatible with the symmetry of the lattice one. If we are exactly at the critical point this greatly simplifies the analysis, since only operators belonging to the $[\sigma]$ family are allowed. Moreover (due to the peculiar null state structure of the spin family) the first quasiprimary operator in the spin family appears at a rather high level and can be neglected in our analysis. Thus as long as we are interested only in the critical point the spin operators on the continuum and on the lattice are simply related by a normalization constant, which we shall call in the following R_σ. The simplest way to obtain R_σ is to look at the analogue of equation (13) on the lattice. This is a well known result [24], which we report here for completeness. The large-distance behaviour of the correlator at the critical point is

$$\langle\sigma_i\sigma_j\rangle_{h=0} = \frac{R_\sigma^2}{|r_{ij}|^{1/4}} \tag{18}$$

where r_{ij} denotes the distance on the lattice between the sites i and j and

$$R_\sigma^2 = \mathrm{e}^{3\xi'(-1)} 2^{5/24} = 0.703\,38\ldots. \tag{19}$$

By comparing this result with equation (13) we find

$$\sigma_l = R_\sigma \sigma = 0.838\,68\ldots\sigma. \tag{20}$$

2.1.4. The lattice Hamiltonian at the critical point. The last step in order to relate the continuum and lattice theories at the critical point is the construction of the lattice Hamiltonian (let us call it H_{lat}) at the critical point. As above, the lattice Hamiltonian will contain all the operators compatible with the symmetries of the continuum one. In this case all the operators belonging to the $[\sigma]$ family are excluded due to the Z_2 symmetry. The operators belonging to the $[\epsilon]$ family are also excluded for a more subtle reason. The Ising model (both on the lattice and in the continuum) is invariant under duality transformations, while the operators belonging to the $[\epsilon]$ family change sign under duality, thus they also cannot appear in $H_{\mathrm{lat}}(t=0)$. Thus we expect

$$H_{\mathrm{lat}} = H_{\mathrm{CFT}} + u_i^0 \int \mathrm{d}^2x\, \eta_i \qquad \eta_i \in [I] \tag{21}$$

where H_{CFT} is the continuum Hamiltonian and the u_i^0 are constants. As mentioned above we can keep in this expansion only the first two terms, which are respectively $T\bar{T}$ and $T^2 + \bar{T}^2$.

2.1.5. The spin–spin correlator on the lattice at the critical point. We have already presented above, in equation (18), the large-distance expansion of the spin–spin correlator on the lattice at the critical point.

Thanks to the exact results obtained in [5–7] (see [25] for a review), we have much more information on this correlator at the critical point. In order to discuss these results let us first introduce a more explicit notation for the spins. We shall denote in the following by $\sigma_{M,N}$ the spin located at the site with lattice coordinates (M, N). By using the translational invariance of the correlator we can always fix one of the two spins at the origin. Thus the most general correlator can be written as $\langle\sigma_{0,0}\sigma_{M,N}\rangle$. A particular role will be played in the following by the correlator along the diagonal, $\langle\sigma_{0,0}\sigma_{N,N}\rangle$, and the one along the axis, $\langle\sigma_{0,0}\sigma_{0,N}\rangle$.

Among the various results on the critical correlators two are of particular relevance for us.

- Remarkably enough, exact expressions exist for the correlator both along the diagonal and along the axis for any value of N. These expressions are rather cumbersome and we shall not report them here. They can be found in [25].
- An asymptotic expansion exists for large values of the separation r_{ij}. For a square lattice, the first three terms have been explicitly evaluated in [8].

This expansion takes the following form (for further details see [8]):

$$\log\langle\sigma_{0,0}\sigma_{M,N}\rangle = \log A - \tfrac{1}{4}\log r + A_1(\theta)r^{-2} + A_2(\theta)r^{-4} + A_3(\theta)r^{-6} + \mathrm{O}(r^{-8}) \tag{22}$$

where

$$\begin{aligned} A_1(\theta) &= 2^{-8}(-1 + 3\cos 4\theta) \\ A_2(\theta) &= 2^{-13}(5 + 36\cos 4\theta + 36\cos 8\theta) \\ A_3(\theta) &= 3^{-1}2^{-19}(-524 - 324\cos 4\theta + 24\,732\cos 8\theta + 28\,884\cos 12\theta) \end{aligned} \tag{23}$$

and

$$\begin{aligned} r^2 &= \tfrac{1}{2}\,(M^2 + N^2) \\ M &= r\,\sqrt{2}\sin\theta \\ N &= r\,\sqrt{2}\cos\theta. \end{aligned} \tag{24}$$

In the following we shall only be interested to the first correction since higher-order corrections are beyond the resolution of our data. Hence we end up with

$$\langle \sigma_{0,0}\sigma_{M,N}\rangle = \frac{A}{r^{1/4}}\big(1 + 2^{-8}(-1 + 3\cos 4\theta)\, r^{-2} + \cdots\big). \tag{25}$$

Remarkably enough this expansion perfectly agrees with what one finds by assuming the presence in the Hamiltonian of the model of the two irrelevant operators discussed in section 2.1.1. In fact both $T\bar{T}$ and $T^2 + \bar{T}^2$ have RG eigenvalue -2, thus, if they are present, they should contribute to the correlator exactly with a term proportional to $1/R^2$. Moreover we expect that the $T^2 + \bar{T}^2$ operator, which has spin 4, should give a term proportional to $\cos(4\theta)$ while the scalar operator $T\bar{T}$ should give a contribution without θ dependence. This is exactly the pattern that we find in equation (25) (see for instance p 218 of [26] for a detailed discussion of this point). However this remarkable agreement also raises a non-trivial problem. In fact the high-precision analyses of [10, 15, 18] clearly exclude the presence in the lattice Hamiltonian of the $T\bar{T}$ operator (while they both confirm that the $T^2 + \bar{T}^2$ is indeed present). It is thus not clear what could be the origin of the scalar term in equation (25). A possible solution to this puzzle is to notice that the results of equation (25) and those of of [10, 15] are obtained with two different choices of the coordinates of the two-dimensional plane.

In fact equation (25) is written in terms of the 'continuous' variable r (which is the one that we must choose if we want to match the lattice results with the continuum limit ones). Both the results of [10] and [15] are instead obtained in the 'lattice reference frame' (which is the most natural variable on the lattice), in which there is no $\sqrt{2}$ when comparing the distances along the axes and along the diagonals. Hence, to compare the correction to the spin–spin correlation function with the findings of [10, 15] we must rewrite (25) in terms of lattice coordinates. This can be easily performed by using the relations (24).

After some algebra we obtain

- axis correlator

$$\langle \sigma_{0,0}\sigma_{0,N}\rangle = \frac{A\, 2^{1/8}}{N^{1/4}}\left(1 + \frac{1}{64}\, N^{-2} + \mathrm{O}(N^{-3})\right) \tag{26}$$

- diagonal correlator

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = \frac{A}{N^{1/4}}\left(1 - \frac{1}{64}\, N^{-2} + \mathrm{O}(N^{-3})\right). \tag{27}$$

These same expansions can also be obtained by looking directly at the exact lattice results for the correlators (see for instance [25], where these $1/N^2$ terms are obtained in full detail).

Looking at equations (26) and (27) we see that with the lattice choice of coordinates the $1/N^2$ term exactly changes its sign as we move from the axis to the diagonal. This is exactly what one would expect for the contribution of a spin-4 operator, and thus it is apparent from equations (26) and (27) that no scalar correction appears at order $1/N^2$, in perfect agreement with the results of [10, 15, 18].

It is only the change of coordinates of equation (24) which induces in the continuum limit a scalar term (which we may well identify in this limit with a $T\bar{T}$ type contribution).

2.2. *Adding the magnetic field*

2.2.1. The continuum theory. The continuum theory in the presence of an external magnetic field is represented by the action

$$\mathcal{A} = \mathcal{A}_0 + h\int \mathrm{d}^2x\, \sigma(x) \tag{28}$$

where $\mathcal{A}_0$ is the action of the conformal field theory.

As a consequence of the applied magnetic field the structure functions acquire an h dependence, so we have in general

$$\Phi_i(r)\Phi_j(0) = \sum_{\{k\}} C_{ij}^{\{k\}}(h, r)\Phi_{\{k\}}(0). \tag{29}$$

Also the mean values of the σ and ϵ operators acquire a dependence on h. Standard renormalization group arguments (for an updated and thorough review on renormalization group theory applied to critical phenomena see [27]) allow one to relate this h dependence to the scaling dimensions of the operators of the theory and lead to the following expressions:

$$\langle\sigma\rangle_h = A_\sigma h^{\frac{1}{15}} + \cdots \tag{30}$$

$$\langle\epsilon\rangle_h = A_\epsilon h^{\frac{8}{15}} + \cdots. \tag{31}$$

The exact value of the two constants A_σ and A_ϵ can be found in [28] and [29] respectively

$$A_\sigma = \frac{2\,\mathcal{C}^2}{15\,(\sin\frac{2\pi}{3} + \sin\frac{2\pi}{5} + \sin\frac{\pi}{15})} = 1.277\,582\,27\ldots \tag{32}$$

with

$$\mathcal{C} = \frac{4\sin\frac{\pi}{5}\Gamma(\frac{1}{5})}{\Gamma(\frac{2}{3})\Gamma(\frac{8}{15})}\left(\frac{4\pi^2\Gamma(\frac{3}{4})\Gamma^2(\frac{13}{16})}{\Gamma(\frac{1}{4})\Gamma^2(\frac{3}{16})}\right)^{\frac{4}{5}} \tag{33}$$

and

$$A_\epsilon = 2.003\,14\cdots. \tag{34}$$

2.2.2. Continuum versus lattice operators. In the presence of a magnetic field the fields belonging to the energy and identity families can also appear in the relation between the lattice and the continuum version of the spin operator. The most general expression is

$$\sigma_l = f_0^\sigma(h_l)\sigma + h_l f_0^\epsilon(h_l)\epsilon + f_i^\sigma(h_l)\sigma_i + h_l f_i^\epsilon(h_l)\epsilon_i + h_l f_i^I(h_l)\eta_i \qquad i \in \boldsymbol{N} \tag{35}$$

where $f_i^\sigma(h_l)$ $f_i^\epsilon(h_l)$ and $f_i^I(h_l)$ are even functions of h_l. By the notation σ_i, ϵ_i and η_i we denote the secondary fields in the spin, energy and identity families respectively. This is the most general expression; however, as a matter of fact, only the first term is relevant for our purposes (all the higher terms give negligible contributions)

$$\sigma_l = R_\sigma\sigma + hp_1\epsilon. \tag{36}$$

The determination of p_1 is rather non-trivial. We shall discuss it in the appendix. It turns out that

$$p_1 \sim 0.0345. \tag{37}$$

It is also important to stress that the lattice and continuum values of the magnetic field do not coincide but are related by

$$h_l = R_h h \tag{38}$$

with $R_h = 1.1923\ldots$ (see [22] for details).

2.2.3. The lattice Hamiltonian. As happened for the conversion from σ to σ_l, also in the construction of the perturbed Hamiltonian on the lattice, several new operators belonging to the energy and spin family must now be taken into account. However it turns out that all the quasiprimary fields of these two families appear at a rather high level and can be neglected in the present analysis. Thus, as far as we are concerned, the only effect of switching on the magnetic field is that the constants in front of the $T\bar{T}$ and $T^2 + \bar{T}^2$ terms in equation (21) acquire an h dependence. For symmetry reasons these must be even analytic functions of h.

2.2.4. The spin–spin correlator. The short-distance behaviour of the spin–spin correlator in the presence of a magnetic field can be obtained by using the so-called IRS approach. A detailed discussion of this approach can be found in [20, 21]; the particular application to the Ising correlators is discussed in [22], to which we refer for details. Here we only list the results. Setting $t \equiv |h|\,|r|^{15/8}$ we have up to $\mathrm{O}(t^2)$

$$\langle\sigma(0)\sigma(r)\rangle|r|^{1/4} = B^1_{\sigma\sigma} + B^2_{\sigma\sigma}t^{8/15} + B^3_{\sigma\sigma}t^{16/15} + \mathrm{O}(t^2) \tag{39}$$

with

$$\begin{aligned} B^1_{\sigma\sigma} &= \hat{C}^{\mathbf{1}}_{\sigma\sigma} = 1 \\ B^2_{\sigma\sigma} &= A_\epsilon \hat{C}^{\epsilon}_{\sigma\sigma} = 1.001\,57\ldots \\ B^3_{\sigma\sigma} &= A_\sigma \widehat{\partial_h C^{\sigma}_{\sigma\sigma}} = -0.515\,81\ldots. \end{aligned} \tag{40}$$

This result holds in the continuum theory. By using the known conversion between continuum and lattice units we can obtain the spin–spin correlator on the lattice:

$$\langle\sigma_l(0)\sigma_l(r)\rangle|r_{ij}|^{1/4} = B^1_{\sigma\sigma,l} + B^2_{\sigma\sigma,l}t_l^{8/15} + B^3_{\sigma\sigma,l}t_l^{16/15} + \mathrm{O}(t_l^2) \tag{41}$$

where r_{ij} is the distance between the two spins on the lattice, measured in units of the lattice spacing, t_l is defined as $t_l \equiv |h_l|\,|r_{ij}|^{15/8}$ and

$$\begin{aligned} B^1_{\sigma\sigma,l} &= 0.703\,384\ldots \\ B^2_{\sigma\sigma,l} &= 0.641\,409\ldots \\ B^3_{\sigma\sigma,l} &= -0.300\,749\ldots. \end{aligned} \tag{42}$$

In the next section we shall see which contributions must be added to this expression as a consequence of the irrelevant fields.

3. Contribution from irrelevant operators to the spin–spin correlator in the presence of a magnetic field

There are two main sources of contributions due to irrelevant operators to equation (41) above. They have different origins and contribute in different ways to the spin–spin correlator. The first correction can be understood as a perturbative contribution to the spin–spin correlator due to irrelevant operators. We shall discuss it in section 3.1. The second correction arises from the new terms in the relation between σ_l and σ, due to the presence of a magnetic field. We shall discuss it in section 3.2.

3.1. Irrelevant operators in the perturbed Hamiltonian

The contributions to the scaling function due to the (possible) presence of additional irrelevant operators in the lattice Hamiltonian can be studied by means of standard perturbative expansions around the CFT fixed point. In order to address the problem in the presence of an external magnetic field we must consider the CFT pertaining to the critical Ising model as perturbed by a mixed relevant/irrelevant perturbation. Hence there will appear correction terms to the scaling behaviour which both depend on the external magnetic field and show a non-trivial (i.e. non-scaling) dependence on the spin–spin distance r_{ij}. Moreover, if the irrelevant operator in which we are interested breaks the rotational invariance (and this is the case for instance of the $(T^2 + \bar{T}^2)$ operator discussed above) we must expect a dependence on the angle θ between the principal axes of the lattice and the direction of the correlator. General symmetry arguments and dimensional analysis strongly constrain these terms. If we keep in our analysis only the first two irrelevant operators $T\bar{T}$ and $(T^2 + \bar{T}^2)$ the most general expression for the scaling function turns out to be

$$\langle\sigma_l(0)\sigma_l(r)\rangle|r_{ij}|^{1/4} = B^1_{\sigma\sigma,l}\left(1 + \frac{-1 + a_1h^2}{|r_{ij}|^2} + \frac{(3 + a_2h^2)\cos(4\theta)}{|r_{ij}|^2}\right)$$
$$+B^2_{\sigma\sigma,l}\left(1 + \frac{b(\theta)}{|r_{ij}|^2}\right)t_l^{8/15} + B^3_{\sigma\sigma,l}\left(1 + \frac{c(\theta)}{|r_{ij}|^2}\right)t_l^{16/15} + \mathrm{O}(t_l^2) \tag{43}$$

where a_1 and a_2 are unknown constants and $b(\theta)$ and $c(\theta)$ are unknown functions of the form

$$b(\theta) = b_1 + b_2\cos(4\theta) \tag{44}$$
$$c(\theta) = c_1 + c_2\cos(4\theta) \tag{45}$$

in the angular variable θ. Let us analyse the above corrections in detail.

Correction to $B^1_{\sigma\sigma,l}$. At the lowest order in the irrelevant coupling, we can identify a scalar correction proportional to $|r_{ij}|^{-2}$ and a spin-4 correction proportional to $\cos(4\theta)\ |r_{ij}|^{-2}$. Both these terms acquire an h dependence due to the magnetic relevant perturbation. The most general expression compatible with the symmetries of the model is

$$B^1_{\sigma\sigma,l}\left(1 + \frac{P_1(h)}{|r_{ij}|^2} + \frac{P_2(h)\cos(4\theta)}{|r_{ij}|^2}\right) \tag{46}$$

where $P_1(h)$ and $P_2(h)$ are even polynomials in the magnetic field h, which in the $h \to 0$ limit must agree with the asymptotic expansion reported in equation (25). Expanding the polynomials up to the first term in h we find

$$\begin{aligned} P_1(h) &= -1 + a_1h^2 \\ P_2(h) &= 3 + a_2h^2 \end{aligned} \tag{47}$$

from which the first term in equation (43) follows. It is easy to see that the terms proportional to a_1 and a_2 are highly suppressed due to the h^2 power. They behave as the $\mathrm{O}(t^2)$ that we have systematically neglected in the previous section. As a matter of fact we have not been able to see these terms in our numerical data and we shall neglect them in the following.

Corrections to $B^2_{\sigma\sigma,l}$ and $B^3_{\sigma\sigma,l}$. Following the same line as discussed above we find in this case

$$\begin{aligned} &B^2_{\sigma\sigma,l}\left(1 + \frac{b(\theta)}{|r_{ij}|^2}\right)t_l^{8/15} \\ &B^3_{\sigma\sigma,l}\left(1 + \frac{c(\theta)}{|r_{ij}|^2}\right)t_l^{16/15} \end{aligned} \tag{48}$$

where $b(\theta)$ and $c(\theta)$ are the most general mixture of scalar and spin-4 terms. They can be expanded in powers of $\cos(4\theta)$. Keeping only the first two orders in the expansion we end up with the expressions reported in equations (44) and (45), where b_1, b_2, c_1 and c_2 are unknown constants.

While the term proportional to $c(\theta)$ cannot be detected within the precision of our data (we shall further discuss this point at the end of the next section), the magnitude of the one proportional to $b(\theta)$ is definitely larger than our numerical uncertainties. Thus we expect to be able to observe such a correction.

3.2. Irrelevant operators in σ_l

We have seen above (see equation (36)) that in the presence of a magnetic field the relation between lattice and continuum spin operator becomes more complicated and a term proportional to $h\epsilon$ appears. Strictly speaking ϵ is not an irrelevant operator; however, this is only an accident: the following terms (those that we neglect) in the correspondence between σ_l and σ would indeed be irrelevant operators. Moreover, exactly as happens for the irrelevant operators, the $h\epsilon$ term also gives a subleading contribution to the spin–spin correlator. For these reasons we have also included this correction in this paper.

Inserting equation (36) in the spin–spin correlator we find

$$\langle\sigma_l(0)\sigma_l(r)\rangle = R_\sigma^2\langle\sigma(0)\sigma(r)\rangle + 2hR_\sigma p_1\langle\sigma(0)\epsilon(r)\rangle + h^2p_1^2\langle\epsilon(0)\epsilon(r)\rangle. \qquad (49)$$

Since we are interested in keeping only terms below $O(t^2)$ the last term in equation (49) can be neglected. Using the known result (see [22] for details)

$$\langle\sigma(0)\epsilon(r)\rangle|r|^{9/8} = B^1_{\sigma\epsilon}t^{1/15} + B^2_{\sigma\epsilon}t + B^3_{\sigma\epsilon}t^{23/15} + O(t^{31/15}) \qquad (50)$$

with

$$B^1_{\sigma\epsilon} = A_\sigma\hat{C}^\sigma_{\sigma\epsilon} = 0.638\,79\ldots$$
$$B^2_{\sigma\epsilon} = \widehat{\partial_h C^{\mathbf{1}}_{\sigma\epsilon}} = 3.296\,27\ldots$$
$$B^3_{\sigma\epsilon} = A_\epsilon\widehat{\partial_h C^\epsilon_{\sigma\epsilon}} = -1.820\,85\ldots$$

we can rewrite equation (49) as

$$\langle\sigma_l(0)\sigma_l(r)\rangle|r|^{1/4} = R_\sigma^2(B^1_{\sigma\sigma} + B^2_{\sigma\sigma}t^{8/15} + B^3_{\sigma\sigma}t^{16/15})$$
$$+2hR_\sigma p_1|r|^{-7/8}(B^1_{\sigma\epsilon}t^{1/15} + B^2_{\sigma\epsilon}t + B^3_{\sigma\epsilon}t^{23/15}). \qquad (51)$$

Only the first term in the second line of equation (51) gives a contribution below $O(t^2)$, thus we end up with

$$\langle\sigma_l(0)\sigma_l(r)\rangle|r|^{1/4} = R_\sigma^2(B^1_{\sigma\sigma} + B^2_{\sigma\sigma}t^{8/15} + B^3_{\sigma\sigma}t^{16/15}) + 2R_\sigma p_1 t|r|^{-22/8}B^1_{\sigma\epsilon}t^{1/15}. \qquad (52)$$

We see that the new term can be considered as a correction, proportional to $\frac{1}{|r|^{22/8}}$, to the $t^{16/15}$ term of equation (41):

$$\langle\sigma_l(0)\sigma_l(r)\rangle|r_{ij}|^{1/4} = B^1_{\sigma\sigma,l} + B^2_{\sigma\sigma,l}t_l^{8/15}$$
$$+(R_\sigma^2B^3_{\sigma\sigma} + 2R_\sigma p_1|r|^{-22/8}B^1_{\sigma\epsilon})/R_h^{16/15}t_l^{16/15} + O(t_l^2). \qquad (53)$$

Inserting the values of the various constants and of p_1 we end up with

$$\langle\sigma_l(0)\sigma_l(r)\rangle|r_{ij}|^{1/4} = B^1_{\sigma\sigma,l} + B^2_{\sigma\sigma,l}t_l^{8/15} + (B^2_{\sigma\sigma,l} + a_3|r|^{-22/8})t_l^{16/15} + O(t_l^2) \qquad (54)$$

with $a_3 = 0.0307\ldots$.

Unfortunately this new contribution is too small to give a reliable signature within the precision of our data. In fact the amplitude of the correction $a_3|r|^{-22/8}t_l^{16/15}$ for distances greater than one lattice spacing is always below (or at most of the order of) 10^{-4} and cannot be disentangled from the other sources of corrections within the precision of our data, which ranges from 10^{-5} to 10^{-4}. The situation is very similar to that of the correction proportional to $c(\theta)$ discussed in the previous section, which in fact has the same t dependence.

However it is interesting to observe that both these corrections are almost on the threshold of observation, and it is perfectly possible that with the next generation of simulation they could be detected. For this reason we also included their analysis in this paper.

4. The simulation

We studied the model with a set of Monte Carlo simulations using a Swendsen–Wang type algorithm, modified so as to take into account the presence of an external magnetic field. We used the same program and simulation setting as [22], to which we refer for a detailed discussion of the performances of the algorithm and of finite-size effects. We simulated the model for 12 different values of the magnetic field, ranging from $h_l = 4.4069 \times 10^{-4}$ to $h_l = 8.8138 \times 10^{-3}$.

For all the values of h_l that we simulated, we studied the correlator $\langle\sigma(0)\sigma(r)\rangle$, for $r = 1, \ldots, 10$, both in the diagonal direction and along the axes. This is an important feature of the present analysis, since it is exactly by comparing the correlator along the axes with the diagonal one that we can extract the spin of the perturbing operators.

For all the simulations the lattice size was chosen to be $L = 200$ (the analysis of [22] shows that for all the values of h_l that we studied this is enough to avoid finite-size effects). Each two measures of the correlators were separated by five SW sweeps. For each single correlator we performed 4×10^5 measures. The values of h_l that we studied are reported in table 1, where we have also listed for completeness the corresponding values of the correlation length ξ. This quantity is very important, since it defines in a quantitative way what we mean by 'short-distance behaviour'. Short distance means 'smaller than the correlation length'. This is in fact the regime in which we expect that the IRS analysis should hold and in which we may hope to observe effects due to the irrelevant operators. ξ is given (in lattice units) by

$$\xi(h_l) = 0.249\,35\ldots h_l^{-\frac{8}{15}} \tag{55}$$

(see [30] for details of this equation and of the continuum to lattice conversion of ξ).

Table 1. The values of h_l (measured in units of β_c) that we studied and the corresponding correlation lengths.

h_l	ξ	h_l	ξ
0.001	15.4	0.007	5.4
0.002	10.6	0.008	5.1
0.003	8.5	0.009	4.8
0.004	7.3	0.01	4.5
0.005	6.5	0.015	3.6
0.006	5.9	0.02	3.1

5. Comparison between predictions and numerical results

Let us start this section by writing the expression that we expect for the spin–spin correlator in a magnetic field, according to the analysis performed in sections 2 and 3 above. We have

$$\begin{aligned}\langle\sigma_l(0)\sigma_l(r)\rangle|r_{ij}|^{1/4} &= \langle\sigma_l(0)\sigma_l(r)\rangle_{h=0}|r_{ij}|^{1/4} + \left(a_1 + \frac{a_2(\theta)}{|r_{ij}|^2} + \cdots\right) t_l^{8/15} \\ &\quad + \left(b_1 + \frac{b_2(\theta)}{|r_{ij}|^2} + \cdots\right) t_l^{16/15} + \mathrm{O}(t_l^2)\end{aligned} \tag{56}$$

where $\langle\sigma_l(0)\sigma_l(r)\rangle_{h=0}|r_{ij}|^{1/4}$ is given by equation (25); $a_1 \equiv B^2_{\sigma\sigma,l}$ and $b_1 \equiv B^3_{\sigma\sigma,l}$, while $a_2(\theta)$ and $b_2(\theta)$ are in principle unknown functions of the angular variable θ.

In this section we shall compare this expression with our numerical estimates for the correlators along the diagonal and the axis. This will allow us to obtain a rather precise estimate of the function $a_2(\theta)$ in the two cases $\theta = 0$ and $\pi/4$. With these two pieces of information we shall also be able to study the spin content of the operators which generate this contribution. In the following we shall confine ourselves to the study of $a_2(\theta)$ since the precision of our data is not sufficient to also extend our analysis to the function $b_2(\theta)$. We performed our analysis in three steps.

(1) As a first step we constructed the combinations

$$G^{\Delta}(r, h_l) \equiv \langle \sigma_l(0)\sigma_l(r)\rangle - \langle \sigma_l(0)\sigma_l(r)\rangle_{h=0} \tag{57}$$

using the fact that the critical point correlator is exactly known. Thus we have

$$G^{\Delta}(r, h_l)|r_{ij}|^{1/4} = \left(a_1 + \frac{a_2(\theta)}{|r_{ij}|^2} + \cdots\right) t_l^{8/15} + \left(b_1 + \frac{b_2(\theta)}{|r_{ij}|^2} + \cdots\right) t_l^{16/15} + \mathrm{O}(t_l^2). \tag{58}$$

(2) Then we study, at a fixed value of r, the h_l dependence of $G^{\Delta}(r, h_l)$. According to equation (58) above we should choose as the fitting function

$$G^{\Delta}(\bar{r}, h_l) = A(\bar{r})h_l^{8/15} + B(\bar{r})h_l^{16/15} \tag{59}$$

(the notation $\bar{r}$ indicates that we are fitting at a fixed value of $|r_{ij}|$). However keeping only the first two terms in the scaling function of the correlator is too rough an approximation. Within the range of values of h_l that we have studied and with the precision that we have obtained, this choice in general leads to very high χ^2 values. Fortunately, thanks to the IRS analysis we know the functional form of the next to leading orders in the h_l expansion of the correlator. It turns out that adding one more term is enough for the correlators ranging from 1 to 7 lattice spacings along both the axis and the diagonal directions, while for the remaining correlators we have to go up to the fourth term in the expansion. The general form of the scaling function at this order turns out to be

$$G^{\Delta}(\bar{r}, h_l) = A(\bar{r})h_l^{8/15} + B(\bar{r})h_l^{16/15} + C(\bar{r})h_l^{30/15} + D(\bar{r})h_l^{32/15} \tag{60}$$

(see [22] for a discussion of this scaling function and the origin of the two new exponents $h_l^{30/15}$ and $h_l^{32/15}$). In this way for all the values of r we found good confidence levels. The values of $A(r)$ that we obtained in this way can be found in table 2. Notice that the data in each fit are completely uncorrelated since they belong to different simulations. In order to give an idea of the quality of the fits we have reported in figure 1 as an example the fit for the value $r = 1$ along the axis.

(3) As a last step we address the r dependence of the function $A(r)$. According to equation (58) we expect the following behaviour:

$$\frac{A(r)}{r^{3/4}} = a_1 + \frac{a_2(\theta)}{r^2}. \tag{61}$$

Fixing a_1 to its known IRS value we end up with a one-parameter fit, which in both cases of $\theta = 0$ and $\pi/4$ has a very good confidence level (see figures 2 and 3).

Our final results are:

- axis correlator

$$a_2(0) = (-0.062 \pm 0.004); \tag{62}$$

- diagonal correlator

$$a_2(\pi/4) = (-0.014 \pm 0.007). \tag{63}$$

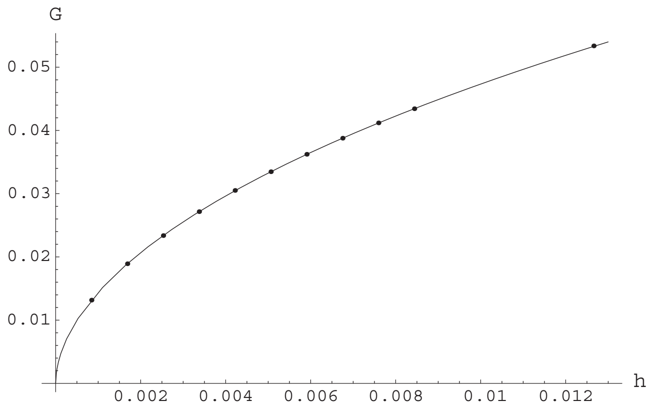

Figure 1. Fit of $G^{\Delta}(\bar{r}, h_l)$ according to equation (60) in the case of the correlator along the axis for $\bar{r} = 1$. The errors of the data are not reported since they are smaller than the symbols.

These results, and in particular the fact that we have different corrections in the two directions (this difference is clearly visible by comparing figure 2 with figure 3), unambiguously show that the contribution is at least partially due to a spin-4 operator. The natural candidate for this role is again the $T^2 + \bar{T}^2$ operator. Our result also suggests that a scalar operator is present in the game. We see two possible reasons for this contribution. The first is that, since we are working in the IRS framework, we are compelled to use the 'continuum limit reference frame'. As we have seen in section 2.1.5 this choice induces a $T\bar{T}$ term and consequently a scalar type correction. The second is that one cannot exclude the presence in the magnetic scaling field of a (rotationally invariant) momentum-dependent term, which would manifest itself exactly as a scalar correction of the type that we observe[5]. This non-trivial possibility has been recently discussed in [16] (see the note on p 8161) to explain the corrections of order t^2 (where t is the reduced temperature) in the second moment of the spin–spin correlator. Most probably both mechanisms are at work in the present case. We cannot distinguish between them within the precision of our data.

Table 2. Values of the first coefficient in equation (60) as a function of the lattice spacing N.

N	$A^{\text{axis}}(N)$	$A^{\text{diag}}(N)$
1	(0.581 ± 0.003)	(0.823 ± 0.003)
2	(1.048 ± 0.004)	(1.396 ± 0.004)
3	(1.446 ± 0.005)	(1.894 ± 0.005)
4	(1.802 ± 0.005)	(2.354 ± 0.007)
5	(2.134 ± 0.006)	(2.777 ± 0.008)
6	(2.448 ± 0.006)	(3.19 ± 0.01)
7	(2.746 ± 0.007)	(3.58 ± 0.01)
8	(3.039 ± 0.009)	(3.96 ± 0.01)
9	(3.316 ± 0.009)	(4.32 ± 0.03)
10	(3.58 ± 0.01)	(4.67 ± 0.04)

[5] We thank A Pelissetto for pointing out to us this interesting possibility.

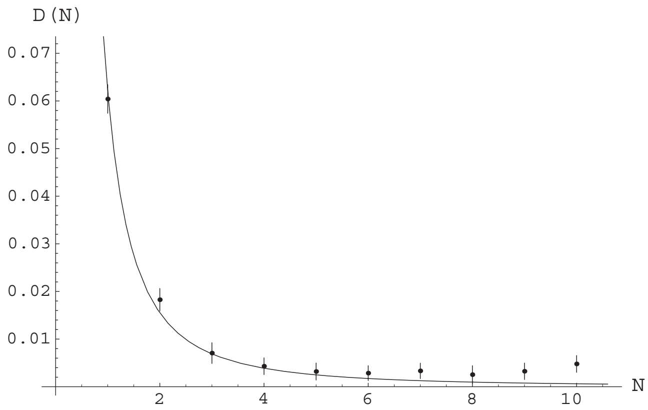

Figure 2. Fit of the difference $D(N) = a_1 - \frac{A(N)}{N^{3/4}}$ in the case of the correlator along the axis. Notice the strong $1/N^2$ correction for small values of N (see equation (61)). For large N the data smoothly converge to the asymptotic value $a_1 = 0.641\,409\ldots$.

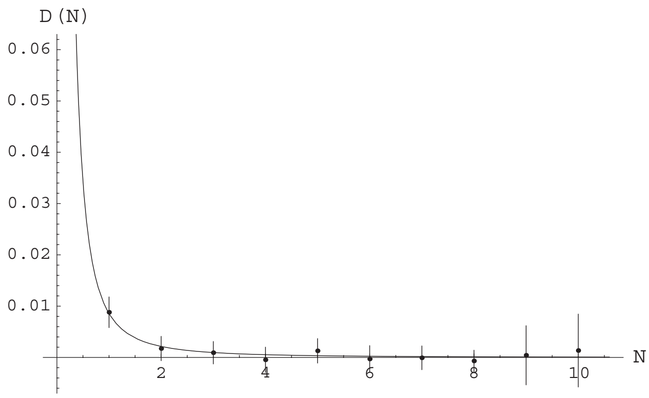

Figure 3. The same as figure 2, but for the correlator along the diagonal. By comparing with figure 2 one can easily see that the $1/N^2$ correction has different magnitudes in the two cases.

6. Concluding remarks

The role of the irrelevant operators in the two-dimensional Ising model has attracted much interest in recent months due to the results on the magnetic susceptibility at $h = 0$ recently reported in [9, 10]. While it is by now clear that contributions due to irrelevant operators are present in the free energy of the Ising model at $h = 0$, nothing is known about the case in which the magnetic perturbation of the critical model is chosen. Moreover the characterization of these irrelevant operators and, possibly, their identification with quasiprimary fields of the Ising CFT is still an open problem. In this paper we tried to make some progress in this direction. We studied the corrections induced by the presence of irrelevant operators in the spin–spin correlator of the two-dimensional Ising model in the presence of an external magnetic field.

Our main results are:

- The $1/r^2$ corrections which are present in the correlator at the critical point also survive unchanged if the magnetic field is switched on. The h-independent part of this contribution is much larger than our statistical errors and can be observed very precisely. It is completely due to the spin-4 irrelevant operator.

- Several new terms appear when the magnetic field is switched on. By using standard perturbative methods one can evaluate the amplitude of the corrections that they induce in the spin–spin correlator. In general these terms are very small. However in one case the amplitude of the expected correction turned out to be definitely larger than our statistical errors. This is the case of the term proportional to

$$\frac{1}{|r|^{1/4}}\left(\frac{t^{8/15}}{r^2}\right) \tag{64}$$

for which we could indeed observe and measure such a correction with good confidence. The comparison between the two values of this correction for the correlator along the diagonal and that along the axis shows that, besides the expected contribution due to the spin-4 $(T^2+\bar{T}^2)$ operator, there is also a scalar term, which may have a twofold origin. This could be due to the $T\bar{T}$ operator, but it could also be the signature of a momentum-dependent term in the magnetic scaling field of the model.

Other contributions due to these or other irrelevant operators are for the moment beyond our resolution, but it may be possible to observe them in the next generation of simulations.

Acknowledgments

We thank M Hasenbusch, A Pelissetto and E Vicari for useful discussions and correspondence on the subject.

Appendix. Determination of p_1

The simplest way to fix the value of p_1 is to look at the expectation value $\langle\sigma_l\rangle$. We expect

$$\langle\sigma_l\rangle = R_\sigma\langle\sigma\rangle + p_1\langle\epsilon\rangle \tag{A.1}$$

from which we have (using the notation of [30])

$$\langle\sigma_l\rangle = R_\sigma A_M h^{\frac{1}{15}} + p_1 A_E h^{\frac{23}{15}} \tag{A.2}$$

from [30] we see that

$$\langle\sigma_l\rangle = \tfrac{16}{15}A_f^l h^{\frac{1}{15}} + \tfrac{38}{15}A_f^l A_{f,2}^l h^{\frac{23}{15}}. \tag{A.3}$$

Using the numerical values reported in [30], i.e.

$$A_f^l = 0.992\,7995\ldots \tag{A.4}$$

$$A_{f,2}^l \sim 0.021\ldots \tag{A.5}$$

we obtain

$$p_1 \sim 0.0345. \tag{A.6}$$

References

[1] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333

[2] Zamolodchikov A B, Zamolodchikov Al B 1979 *Ann. Phys., NY* **120** 253
For a review see for instance Mussardo G 1992 *Phys. Rep.* **218** 215

[3] Zamolodchikov A B 1989 *Adv. Stud. Pure Math.* **19** 641
Zamolodchikov A B 1988 *Int. J. Mod. Phys.* A **3** 743

[4] Smirnov F A 1992 *Form Factors in Completely Integrable Models of Quantum Field Theory* (Singapore: World Scientific)
Cardy J L and Mussardo G 1993 *Nucl. Phys.* B **410** [FS] 451
[5] Kaufman B and Onsager L 1949 *Phys. Rev.* **76** 1244
[6] Fisher M E and Burford R J 1967 *Phys. Rev.* **156** 583
[7] McCoy B M, Wu T T, Tracy C A and Barouch E 1976 Two-dimensional Ising model: exact theory in the scaling region *Phys. Rev.* B **13** 316
[8] Au-Yang H and Perk J H H 1984 *Phys. Lett.* A **104** 131
[9] Nickel B 1999 *J. Phys. A: Math. Gen.* **32** 3889
Nickel B 2000 *J. Phys. A: Math. Gen.* **33** 1693
[10] Orrick W P, Nickel B, Guttmann A J and Perk J H H 2001 The susceptibility of the square lattice Ising model: new developments *J. Stat. Phys.* **102** 795
(Orrick W P, Nickel B, Guttmann A J and Perk J H H 2001 *Preprint* cond-mat/0103074)
Orrick W P, Nickel B, Guttmann A J and Perk J H H 2001 Critical behaviour of the two-dimensional Ising susceptibility *Phys. Rev. Lett.* **86** 4120
(Orrick W P, Nickel B, Guttmann A J and Perk J H H 2000 *Preprint* cond-mat/0009059)
[11] Warnaar S O, Pearce P A, Seaton K A and Nienhuis B 1994 *J. Stat. Phys.* **74** 469
[12] Delfino G 1998 *Phys. Lett.* B **419** 291
Delfino G and Simonetti P 1996 *Phys. Lett.* B **383** 450
Delfino G and Mussardo G 1995 *Nucl. Phys.* B **455** 724
[13] Warnaar S O, Nienhuis B and Seaton K A 1992 *Phys. Rev. Lett.* **69** 710
Bazhanov V V, Nienhuis B and Warnaar S O 1994 *Phys. Lett.* B **322** 198
Grimm U and Nienhuis B 1997 *Phys. Rev.* E **55** 5011
[14] Batchelor M T and Seaton K A 1998 *Nucl. Phys.* B **520** 697
Batchelor M T and Seaton K A 1997 *J. Phys. A: Math. Gen.* **30** L479
Seaton K A and Batchelor M T 2001 The inversion relation and the dilute $A_{3,4,6}$ eigenspectrum *J. Stat. Phys.* **102** 1019
(Seaton K A and Batchelor M T 2000 *Preprint* hep-th/0005100)
[15] Caselle M, Hasenbusch M, Pelissetto A and Vicari E 2001 Irrelevant operators in the 2d Ising model *Preprint* cond-mat/016372
[16] Calabrese P, Caselle M, Celi A, Pelissetto A and Vicari E 2000 *J. Phys. A: Math. Gen.* **33** 8155
[17] Aharony A and Fisher M E 1983 *Phys. Rev.* B **27** 4394
[18] Reinicke P 1987 *J. Phys. A: Math. Gen.* **20** 4501
Reinicke P 1987 *J. Phys. A: Math. Gen.* **20** 5325
[19] Gartenhaus S and McCullough W S 1988 *Phys. Rev.* B **38** 11688
[20] Guida R and Magnoli N 1996 *Nucl. Phys.* B **471** 361
[21] Guida R and Magnoli N 1997 *Nucl. Phys.* B **483** 563
[22] Caselle M, Grinza P and Magnoli N 2000 *Nucl. Phys.* B **579** 635
[23] Henkel M 1999 *Conformal Invariance and Critical Phenomena* (Berlin: Springer)
[24] Wu T T 1966 *Phys. Rev.* **149** 380
[25] McCoy B M and Wu T T 1973 *The Two Dimensional Ising Model* (Cambridge, MA: Harvard University Press)
McCoy B M 1995 *Statistical Mechanics and Field Theory* ed V V Bazhanov and C J Burden (Singapore: World Scientific)
[26] Cardy J L 1989 *Conformal Invariance and Statistical Mechanics Field, Strings and Critical Phenomena (Les Houches Session XLIX)* ed E Brezin and J Zinn-Justin (Amsterdam: Elsevier)
[27] Pelissetto A and Vicari E 2000 Critical phenomena and renormalization-group theory *Preprint* cond-mat/0012164
[28] Fateev V 1994 *Phys. Lett.* B **324** 45
[29] Fateev V, Lukyanov S, Zamolodchikov A and Zamolodchikov Al 1998 *Nucl. Phys.* B **516** 652
[30] Caselle M and Hasenbusch M 2000 *Nucl. Phys.* B **579** 667